Robust Determination of Rotation-Angles for Closed Regions using Moments

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Abstract

In this paper we introduce a new moment based approach for determining the rotation angle between two closed regions. It is wideley in use to determine the rotation angle by the second order moments. However, in these cases if the inertial ellipse degenerates to the inertial circle, then this method fails. In this paper we generalize that to the third and fourth order moments and show that the method is very robust in most cases.
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1 Introduction

Matching problems are very important in practical image processing. In a lot of practical applications only the rotation angle is of interest. In such a case a simple and efficient method is the state of the art. Such a simple method is using the orientation of the inertial ellipse for a given closed region, see [1, 2] . The inertial ellipse is based on the second order area moments of the object. If this inertial ellipse degenerates to an inertial circle the method fails. In the present paper we generalize this fact to "inertial objects" using higher order moments. We show, that we can use also the third or fourth order moments in the case of a degenerating inertial ellipse.

2 The Well-known Moment Based Method

- First, we normalize the translation by the centroid, i.e. $m_{10} = m_{01} = 0$ and m_{ij} are the central moments.

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- Second, we normalize the rotation by the constraint $m_{11} = 0$, and we get for the angle φ:
$$\tan 2\varphi \;=\; \frac{2m_{11}}{m_{20} - m_{02}} \;. \qquad (1)$$
This is the well-known principal axes transform of the inertial ellipse. This solution is unique exept for a period of $\frac{\pi}{2}$. The angle may be with respect to either the major principal axis or the minor principal axis. A usual way to determine an unique orientation is to set additional constraints, see e.g. [4]. If the conditions $|m_{20} - m_{02}| \leq \epsilon$ and $|m_{11}| \leq \epsilon$ are satisfied (ϵ small) the method fails.

The derivation of the inertial ellipse of an object is very simple:

- We transform the moment m_{20} depending of the rotation angle φ:
$$m_{20}(\varphi) = m_{20}\cos^2\varphi \;+\; 2m_{11}cos\varphi sin\varphi \;+\; m_{02}sin^2\varphi \;\;. \qquad (2)$$
This is the inertial function of the object, which can be taken by a parametrization of the object with respect to the rotation angle φ. The minimum of the function (2) defined by one of the roots using
$$m_{11} + (m_{20} - m_{02})\tan\varphi - m_{11}\tan^2\varphi \;=\; 0 \qquad (3)$$
is the expression (1). However, the expression (1) is also a solution for the maximum of the function (2). Thus we have prolems with the period $\frac{\pi}{2}$. Consequently, we define as an unique orientation with the period π the minimum of the inertial function $m_{20}(\varphi)$ according to (2). If any object has a symmetry with a rotation period lower than π then follows $m_{20}(\varphi) = constant$.
- We substitute $\sin(\varphi) \;=\; -\frac{x}{\sqrt{m_{20}}}$, $\cos(\varphi) \;=\; \frac{y}{\sqrt{m_{20}}}$ in expression (2) and receive
$$1 \;=\; m_{20}y^2 \;-\; 2m_{11}xy \;+\; m_{02}x^2 \;, \qquad (4)$$
the well-known equation for the inertial ellipse.

3 Objects of a Degenerated Inertial Ellipse

It is not correct to believe that all objects with a degenerated inertial ellipse have especial symmetries with a rotation period lower than π. For these purposes we choose any object and calculate the central moments. Now we transform the moments and the object (similar to the whitening transform), see [5] :

1. It is to normalize a x-shearing $x' = x + \gamma y, y' = y$ by $m'_{11} = 0$. It is $\gamma = -\frac{m_{11}}{m_{02}}$ and now transform all moments to $m'_{pq} = \sum_{k=0}^{p}\binom{p}{k}\gamma^{p-k}m_{k,p+q-k}$.

2. It is to normalize an anisotrope scaling $x'' = \alpha x', y'' = \beta y'$ by $m''_{20} = 1, m''_{02} = 1$. It is $\alpha = \sqrt[8]{\frac{m_{02}}{m_{20}^3}}, \beta = \sqrt[8]{\frac{m_{20}}{m_{02}^3}}$ and now transform all moments to $m''_{pq} = \alpha^{p+1}\beta^{q+1}m'_{pq}$.

3. With the calculated complete transformation we transform the given object to an object which has a degenerate inertial ellipse.

In the Fig. 1 it is displayed the original object (left side) which has an inertial ellipse and the transformed object (right side, thought has been given to an additional isotrope scaling for displaying of the normalized object) which has an inertial circle. It can be seen that there are no especial symmetries of the object. For these objects with an inertial circle we try to use higher order moments to meet the requirements for a simple determination of the orientation using closed regions.

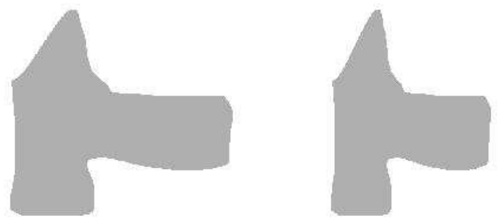

Figure 1: Left side: Object has an inertial ellipse, Right side: Transformed object has an inertial circle

4 Generalization to Higher Order Moments

First of all, we generalize the inertial function (2) to higer order moments:

$$m_{n,0}(\varphi) \;=\; \sum_{k=0}^{n}\binom{n}{k} m_{n-k,k}\cos^{n-k}\varphi\sin^{k}\varphi \;,\; n = 2,3,4,5,6,7,.... \qquad (5)$$

This is a general moment based parametrization for the contour of the object. If an object is rotated, the both functions are shifted against each other. The shift can be determined by special landmarks of the function (e.g. the minimum) or an matching process. Second, we substitute $\sin\varphi = -\frac{x}{\sqrt[n]{m_{n,0}}}, \cos\varphi = \frac{y}{\sqrt[n]{m_{n,0}}}$ (n even), in the expression (5) and obtain

$$1 \;=\; \sum_{k=0}^{n}\binom{n}{k}(-1)^k m_{n-k,k}y^{n-k}x^k \;. \qquad (6)$$

This generalization concerning the inertial objects makes sense only for an even order of the moments. The idea for a practical determination of the orientation of an object ist the following:

- Begin with $n = 2$ and determine the minimum of function $m_{n,0}(\varphi)$. This is the orientation of the inertial object and can be used as the orientation of the object.
- If $m_{n,0}(\varphi) = const$ holds, then the inertial object is a circle. Try to use the function $m_{n+1,0}(\varphi)$. If this function is also constant, then put $n := n+2$ and repeat the process.

For practical applications the moments exceeding fourth order are too sensitive to noise. For that reason, we analyze the case $n = 3, 4$ in the following.

5 Using Moments up to Fourth Order

The moments up to fourth order result in the inertial functions

$$m_{3,0}(\varphi) = m_{30}\cos^3\varphi + 3m_{21}\cos^2\varphi\sin\varphi + 3m_{12}\cos\varphi\sin^2\varphi + m_{03}\sin^3\varphi \ , \quad (7)$$

$$\begin{aligned} m_{4,0}(\varphi) = \ & m_{40}\cos^4\varphi + 4m_{31}\cos^3\varphi\sin\varphi + 6m_{22}\cos^2\varphi\sin^2\varphi + \\ & +4m_{13}\cos\varphi\sin^3\varphi + m_{04}\sin^4\varphi \end{aligned} \ , \quad (8)$$

and in the inertial object described by the polynom of fourth order

$$1 = m_{40}y^4 - 4m_{31}y^3x + 6m_{22}x^2y^2 - 4m_{13}xy^3 + m_{04}x^4. \quad (9)$$

In generalization to (3) we must find the minimum by finding the roots of the third order polynomial

$$m_{21} + (2m_{12} - m_{30})\tan\varphi + (m_{03} - 2m_{21})\tan^2\varphi - m_{12}\tan^3\varphi = 0 \quad (10)$$

or the fourth order polynomial

$$\begin{aligned} & m_{31} + (3m_{22} - m_{40})\tan\varphi + (3m_{13} - 3m_{31})\tan^2\varphi + \\ & +(m_{04} - 3m_{22})\tan^3\varphi - m_{13}\tan^4\varphi = 0 \ . \end{aligned} \quad (11)$$

In Fig. 2 an object can be seen, which has an inertial circle using the second order moments, but the displayed inertial object of fourth order (9) has an orientation. In Fig. 3 the function $m_{40}(\varphi)$ has a typical miminum with a period π, the function $m_{30}(\varphi)$ has also a typical minimum even with a period 2π.

In Fig. 4 it is displayed a square which has a oriented inertial object of fourth order. In Fig. 5 the inertial function of fourth order is displayed and has a period of $\frac{\pi}{2}$. All third order moments vanish, this means that $m_{30}(\varphi) \equiv 0$. If any object has a symmetry with a rotation period lower than $\frac{\pi}{2}$ then follows $m_{40}(\varphi) = const$. This means that the minimum of $m_{40}(\varphi)$ as the orientation of fourth order is unique with the period π or $\frac{\pi}{2}$. With respect to the inertial object of fourth order the question arises: Why we do not use the moments $m_{04}(\varphi)$ or $m_{22}(\varphi)$? It is equivalent to use $m_{04}(\varphi)$, this function has only a phase shift of $\frac{\pi}{2}$ compared with $m_{04}(\varphi)$ and has the same period π. However, the function $m_{22}(\varphi)$ is not adequate simply, because this function has only a period of $\frac{\pi}{2}$.

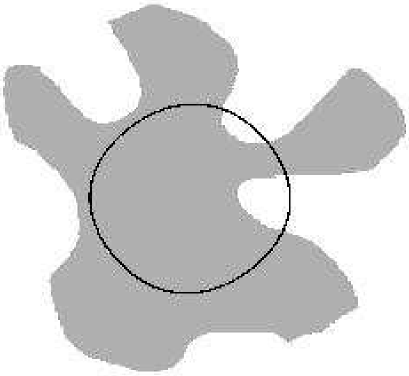

Figure 2: Object has an inertial circle, but an oriented inertial object of fourth order

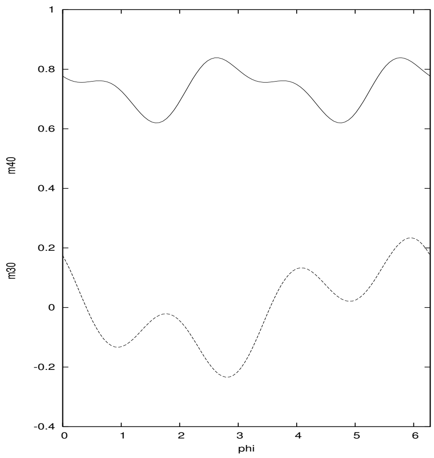

Figure 3: $m_{30}(\varphi)$ and $m_{40}(\varphi)$

6 Criterion for Using Third and Fourth Order Moments

Now we have to decide with the help of a numerical criterion whether the second order moments work or not. This criterion must be normalized and invariant with the respect to the rotation. With the help of the simple inquality

$$m_{11}^2 \;\leq\; m_{20}m_{02}$$

it can be proved the following normalization

$$0 \;\leq\; m_{2norm} = \frac{(m_{20}-m_{02})^2+4m_{11}^2}{(m_{20}+m_{02})^2} \;\leq\; 1 \;. \tag{12}$$

The numerator $H_2 = (m_{20}-m_{02})^2 + 4m_{11}^2$ and the root of the denominator $H_1 = m_{20} + m_{02}$ are well-known Hu-invariants, see [3]. Therefore, the measure m_{2norm} is rotation invariant. Is $m_{2norm} = 0$, then we have exactly an object with an inertial circle. Is $m_{2norm} \approx 1$ then the object has a very prolate inertial ellipse. Let be a the length of the semi-major axis and b the length of the semi-minor axis concerning the inertial ellipse of an object. Let be $s = \frac{a}{b}$ the ratio of both semi-axes, then we get

$$\sqrt{m_{2norm}} \;=\; \frac{s^2-1}{s^2+1}\;,$$

and we can choose the threshold depending on the ratio s. For a ratio of $\frac{11}{10}$ we get a safe threshold with $m_{2norm} = 0.05$. It follows, if $m_{2norm} \geq 0.05$ then

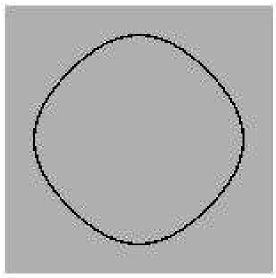

Figure 4: An ideal square has an oriented inertial object of fourth order

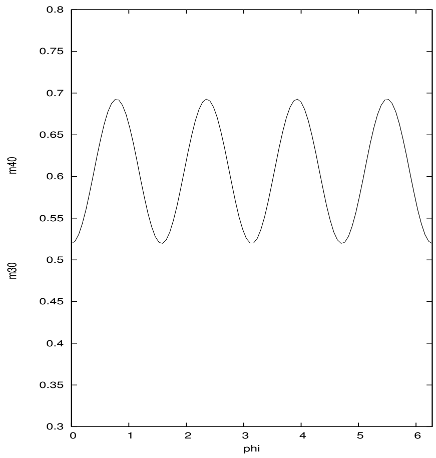

Figure 5: $m_{40}(\varphi)$ ($m_{30}(\varphi)$ vanishes)

we use the second order moments instead of the third or fourth order moments. Concerning the sensitivity to noise we firstly consider the third order moments. The method fails if nearly $m_{30}(\varphi)$ is constant. A good criterion is a moment-based Hu-invariant:

$$m_{3norm} \;=\; \sqrt{(m_{30} - 3m_{12})^2 + (3m_{21} - m_{03})^2} \;\;.$$

If m_{3norm} is nearly 0, the method fails. In this case we use the fourth order moments and consider also the case if $m_{40}(\varphi)$ is constant. E.g. the conditions $m_{40} = m_{04} = 3m_{22}$ and $m_{31} = m_{13} = 0$ imply that situation. This means that the inertial object of fourth order is also a circle. Now we have to define a measure similar m_{2norm} which is normalized and rotation invariant. For this purpose we develop Hu-invariants using fourth order moments:

$$m_{4norm} \;=\; \frac{(m_{40} - m_{04})^2 + 4(3m_{22} - m_{40})(3m_{22} - m_{04}) + 16(m_{31} - m_{13})^2}{(m_{40} + 2m_{22} + m_{04})^2} \;.$$

For the derivation of the fourth order Hu-invariants it can be used complex-moments, see [1, 6].

7 The Algorithm

- For a given closed region compute the central moments up to fourth order.
- For numerical reasons normalize all moments to $m'_{00} = 1$ by $m'_{p,q} = (\frac{1}{\sqrt{m_{00}}})^{p+q+2} m_{p,q}$.
- If $m_{2norm} \geq threshold_2$ compute the angle from the second order moments using (1) STOP.

- If $m_{2norm} < threshold_2$ then compute m_{3norm}. If $m_{3norm} \geq threshold_3$ compute the minimum of the inertial function of third order (7) with period 2π. This can be done directly or by finding the roots of (10) using that root with a minimum value. STOP.
- If $m_{3norm} < threshold_3$ then compute m_{4norm}. If $m_{4norm} \geq threshold_4$ compute the minimum of the inertial function of fourth order (8) with period π. This can be done directly or by finding the roots of (11) using the root with a minimum value. STOP.
- If $m_{4norm} \leq threshold_4$ the method fails. STOP.

8 Numerical Experiments

It is trivial that the fourth order moments are more sensitive to noise than the third order moments. However, our problem is the sensitivity of the angle-determination against noise of the contour and that problem is not the same one. The experiments have been carried out in the following way:

- Generate any closed region, compute the moments and transform the object so that the object has a degenerate inertial ellipse, see section 3. This means that the second order moments fail for a determination of the orientation of the normalized object. Additionally, normalize the moments to $m'_{00} = 1$, see section 7.
- Choose randomly any rotation-angle, transform the object and choose noise for every pixel in dependence of a given standard deviation. Compute the moments of this object and normalize the moments to $m'_{00} = 1$.
- Compute the orientation of both objects and calculate the difference to the exact rotation angle.

Our main result is that the robustness basically depends on the shape of the given closed region. In a lot of experiments we have detected the optimal thresholds by $m_{2norm} = 0.05$, $m_{3norm} = 0.01$, and $m_{4norm} = 0.1$. The third order moments are sensitive to symmetries of the object. If there are nearly symmetries, then holds $m_{3norm} \leq 0.01$. We consider Fig. 6, it is displayed an object which has no symmetries, it is $m_{3norm} = 0.04$ and $m_{4norm} = 0.001$. We see that the third order moments work very well and are robust against noise. In Fig. 8 it is given an object nearly with symmetries $m_{3norm} = 0.004$, but $m_{4norm} = 0.13$. This means that the third order moments are instable, but the fourth order moments are stable and not sensitive to noise. In Fig. 10 an object is given with a high stability of the fourth order moments indicated by $m_{4norm} = 0.8$. The object has no symmetries $m_{3norm} = 0.05$ and therefore both, the third order and the fourth order moments, work very well in dependence of noisy contours.

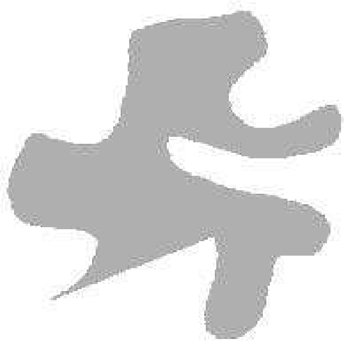

Figure 6: Object with $m_{3norm} = 0.04, m_{4norm} = 0.001$

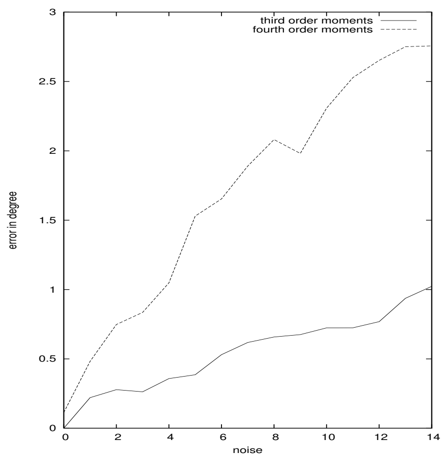

Figure 7: $m_{30}(\varphi)$ is stable, m_{40} is unstable

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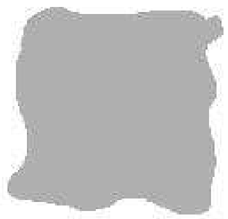

Figure 8: Object with $m_{3norm} = 0.004, m_{4norm} = 0.13$

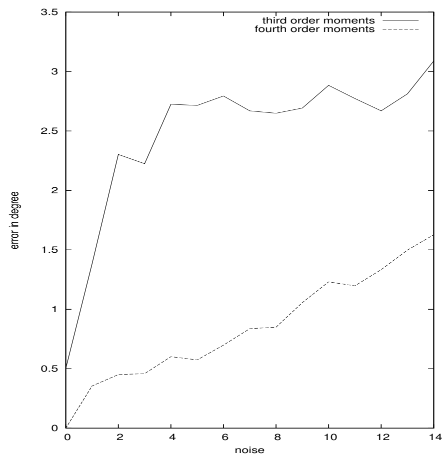

Figure 9: $m_{30}(\varphi)$ is unstable, $m_{40}(\varphi)$ is stable

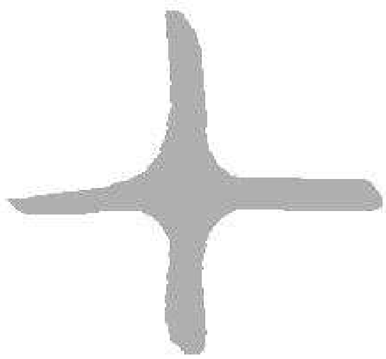

Figure 10: Object with $m_{3norm} = 0.05, m_{4norm} = 0.8$

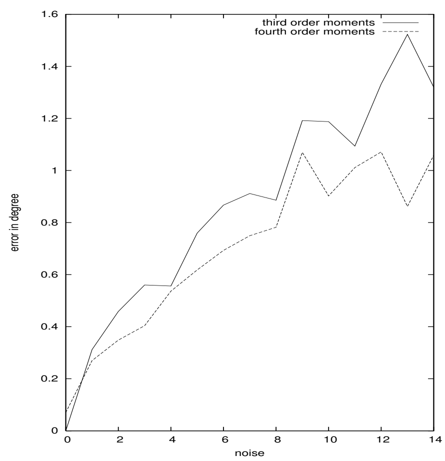

Figure 11: $m_{30}(\varphi)$ and $m_{40}(\varphi)$ are stable