Bias Removal Algorithm for 2-D Equation Error Adaptive IIR Filters

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Abstract. This paper proposes a bias removal algorithm for equation error-based 2-D adaptive cascade IIR filters with separable denominator function. As well known, equation error-based adaptive IIR filtering algorithms have the advantages of fast convergence and unimodal mean-square-error surface. These advantages, however, come along with the drawback of biased parameter estimates in the presence of measurement noise. The adaptive filter structure in the proposed algorithm is based on the concept of backpropagating the desired signal through a cascade of the denominator vertical and horizontal sections. To handle the bias problem, the proposed algorithm uses a scaled value of the output error of each of the cascaded sections as an estimate for the measurement noise embedded in the signal part of the coefficient-update procedure of that section. Thus, while maintaining the advantages of easy stability monitoring, fast convergence, and low computational load, the effect of the measurement noise is suppressed. Input-Output stability analysis is carried out, and the constraints required to maintain stability are derived. Simulation examples are presented to support the effectiveness and the usability of the proposed bias removal algorithm in 2-D system identification and image enhancement applications.

Key Words: 2-D equation error IIR filter, backpropagation adaptive filter, LMS algorithm

1. Introduction

Equation error adaptive IIR filters have the advantages of fast convergence and unimodal mean square error surface. However, their main drawback is that they converge to biased parameter estimates in the presence of measurement noise.

A 1-D equation error cascade IIR filtering algorithm has been proposed by Gao and Snelgrove [1]. This algorithm is based on the concept of backpropagating the desired signal through the inverse of the all pole second order sections such that new intermediate errors are generated; then the filter coefficients are adapted to minimize the intermediate errors. This cascade structure has the advantages of easy stability check and low parameter sensitivity. Moreover, minimizing the intermediate equation error functions instead of the output error offers significant reduction in the gradients' computational complexity [1]. Such advantages are of great interest in 2-D adaptive IIR filters with applications to image enhancement and 2-D system identification.

Toshima et al. [2] have proposed a direct extension of the 1-D backpropagation concept [1] to 2-D case for separable denominator adaptive IIR filters. In addition to the simplicity of stability monitoring, 2-D IIR filters with separable denominator function offer signifi-
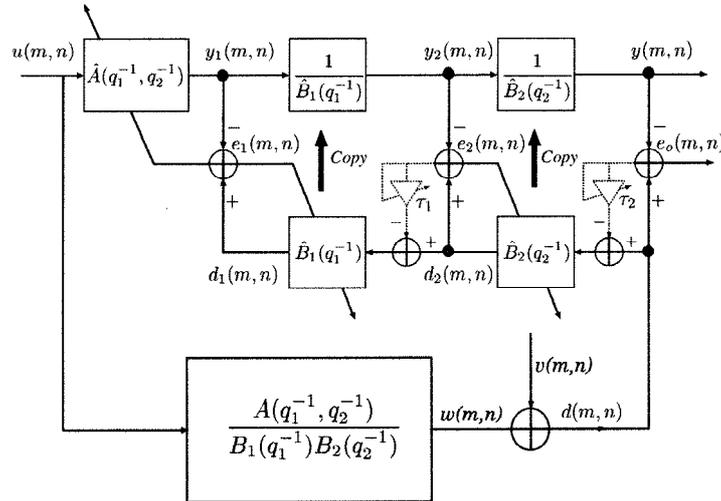
Figure 1. Equation error formulation for separable denominator 2-D IIR filters.

cant reduction in the computational load when compared with direct form 2-D IIR filters. Moreover, since the numerator is nonseparable polynomial, they can be efficiently used to approximate nonseparable 2-D IIR filters [3]. However, the drawback of the backpropagation cascade structures (1-D and 2-D case) is that they are based on minimizing equation error functions; accordingly, it is expected that they converge to biased parameter estimates when the desired signal is contaminated with measurement noise [4]. So far, stability analysis and the performance of the 1-D backpropagation cascade structure [1] as well as the 2-D one [2] have not been considered in the literature.

In this paper we propose a 2-D Bias Removal Algorithm (2DBRA) for the backpropagation cascade structure [2]. The proposed bias removal technique is based on the idea of using a scaled value of the output error as an estimates for the measurement noise. This idea is adopted from the bias remedy LMS (BRLMS) algorithm proposed by Lin and Unbehauen [5] for direct form IIR filter. However, in this paper, we suggest to apply the output-error feedback at the output of each of the cascaded sections as shown in Figure 1. Thus, while maintaining the advantages of easy stability monitoring, fast convergence, and low computational load, the effect of the measurement noise is suppressed.

Through our stability analysis of the the proposed algorithm we show that there are two possible reasons for the convergence of the algorithm [2] to biased parameter estimates. The first one is due to the cascade connection; the second is due to the measurement noise in the desired signal. The effect of the latter can be counteract by introducing the output-error feedback in the 2DBRA, while it is necessary to use small step size parameters to reduce the effect of the cascade connection.

Note that a bias removal technique and performance analysis similar to those presented in this paper can be applied to the 1-D cascaded structure [1].

After introducing the the 2DBRA, stability analysis of the proposed algorithm is discussed. System identification and image enhancement experimental results are then presented to demonstrate the effectiveness and the superiority of the 2DBRA over the 2-D adaptive algorithm [2].

2. 2-D Bias Removal Algorithm (2DBRA)

Consider the 2-D stable separable denominator IIR filter

$$w(m, n) = \sum_{i=0}^{N_1} \sum_{j=0}^{N_2} a(i, j) u(m - i, n - j) + \sum_{\substack{i=0 \\ (i,j)\neq(0,0)}}^{M_1} \sum_{j=0}^{M_2} b_1(i) b_2(j) w(m - i, n - j); \quad (1)$$

$$d(m, n) = w(m, n) + v(m, n) \quad (2)$$

where $u(m, n), m = 0, \ldots, M, n = 0, \ldots, N$, is the input signal, $d(m, n)$ is the observable output signal, and $v(m, n)$ is a zero mean measurement noise assumed to be independent of $w(m, n)$ and $u(m, n)$.

Let q_1^{-1} and q_2^{-1} denote spatial delay operators in the horizontal and vertical directions respectively. For the separable denominator adaptive IIR filter shown in Figure 1, the input signal $u(m, n)$ is passed through the adaptive filter's transversal section ($\hat{A}(q_1^{-1}, q_2^{-1}) = \sum_{i=0}^{N_1} \sum_{j=0}^{N_2} \hat{a}(i, j) q_1^{-i} q_2^{-j}$), while the desired signal $d(m, n)$ is backpropagated through the cascade of the adaptive filter's denominator vertical and horizontal sections $\hat{B}_2(q_2^{-1}) = 1 - \sum_{j=1}^{M_2} \hat{b}_2(j) q_2^{-j}$, and $\hat{B}_1(q_1^{-1}) = 1 - \sum_{i=1}^{M_1} \hat{b}_1(i) q_1^{-i}$ respectively. Then two intermediate error functions, namely $e_1(m, n)$ and $e_2(m, n)$, are generated as follows:

$$e_1(m, n) = d_1(m, n) - y_1(m, n)$$
$$= d_2(m, n) - \hat{\theta}_1^T (k - 1) \varphi_1(m, n) \quad (3)$$
$$e_2(m, n) = d_2(m, n) - y_2(m, n)$$
$$= \hat{\theta}_2^T (k - 1) \varphi_2(m, n) - y_2(m, n). \quad (4)$$

In the inner product notations, k denotes the iteration number used in updating the coefficients; the value of k is some function of (m, n) specifying the indexing scheme, and

$$\hat{\theta}_1(k) = [\hat{b}_1^k(1), \ldots, \hat{b}_1^k(M_1), \hat{a}^k(0, 0), \ldots, \hat{a}^k(N_1, N_2)]^T \quad (5)$$
$$\varphi_1(m, n) = [d_2(m - 1, n), \ldots, d_2(m - M_1, n),$$
$$u(m, n), \ldots, u(m - N_1, n - N_2)]^T \quad (6)$$
$$\hat{\theta}_2(k) = [1, -\hat{b}_2^k(1), \ldots, -\hat{b}_2^k(M_2)]^T \quad (7)$$
$$\varphi_2(m, n) = [d(m, n), d(m, n - 1), \ldots, d(m, n - M_2)]^T. \quad (8)$$

To handle the bias problem, the 2DBRA tries to counteract the effect of the measurement noise $v(m, n)$ in the desired signal $d(m, n)$ by using a scaled value of the output-error

$e_o(m, n) = d(m, n) - y(m, n)$ as an estimate for the measurement noise $v(m, n)$. The filter parameter vector $\hat{\boldsymbol{\theta}}_2(k)$ is then updated subject to minimizing the intermediate error function $e_2(m, n)$ as follows:

$$\hat{\boldsymbol{\theta}}_2(k) = \hat{\boldsymbol{\theta}}_2(k - 1) - \boldsymbol{\mu}_\Theta e_2(m, n)[\boldsymbol{\varphi}_2(m, n) - \tau_2(m, n)\varepsilon_o(m, n)] \tag{9}$$

where $\boldsymbol{\mu}_\Theta = \text{diag}[0, \mu_2, \ldots, \mu_2]$ is a diagonal step size matrix, $\varepsilon_o(m, n)$ is the output error vector given by

$$\varepsilon_o(m, n) = [e_o(m, n), e_o(m, n - 1), \ldots, e_o(m, n - M_2)]^T \tag{10}$$

and $\tau_2(m, n)$ is a scaling factor defined as

$$\tau_2(m, n) = \min\left(1, \alpha_2 \frac{\|\boldsymbol{\varphi}_2(m, n)\|}{\|\varepsilon_o(m, n)\|}\right), \quad \alpha_2 \geq 0 \tag{11}$$

with $\|\ \ \|$ denoting the Euclidean norm.

In a similar way, the 2DBRA uses a scaled value of the error signal $e_2(m, n)$ to estimate the colored noise $v_1(m, n) = (1 - \sum_{j=1}^{M_2} \hat{b}_2^{k-1}(j)q_2^{-j})v(m, n)$, that reaches the intermediate signal $d_2(m, n)$ (see Figure 1). The filter parameter vector $\hat{\boldsymbol{\theta}}_1(k)$ is then updated subject to minimizing the intermediate error function $e_1(m, n)$ as follows:

$$\hat{\boldsymbol{\theta}}_1(k) = \hat{\boldsymbol{\theta}}_1(k - 1) + \mu_1 e_1(m, n)[\boldsymbol{\varphi}_1(m, n) - \tau_1(m, n)\varepsilon_2(m, n)] \tag{12}$$

where

$$\varepsilon_2(m, n) = [e_2(m - 1, n) \ldots e_2(m - M_1, n), \mathbf{o}]^T \tag{13}$$

and $\tau_1(m, n)$ is a scaling factor defined as

$$\tau_1(m, n) = \min\left(1, \alpha_1 \frac{\|\boldsymbol{\varphi}_1(m, n)\|}{\|\varepsilon_2(m, n)\|}\right), \quad \alpha_1 \geq 0. \tag{14}$$

The time varying scaling factor $\tau_1(m, n)(\tau_2(m, n))$ defined in Eq. (14)((11)) is chosen to be inversely proportional to the variance of the error $e_2(m, n)(e_o(m, n))$ such that output-error feedback mechanism works only when the output-error $e_2(m, n)(e_o(m, n))$ becomes good estimates of the measurement noise in $d_2(m, n)$ $(d(m, n))$.

3. Stability Analysis of the 2DBRA

Rigorous convergence analysis of the 2-D cascade structure in Figure 1 as a whole dynamic is very complicated due to the interaction between the cascaded sections. In an attempt to simplify the analysis of the dynamical behavior of the 2DBRA, we divide this structure into two parts. The first part consists of the transversal section and the denominator horizontal section with input vector $\boldsymbol{\varphi}_1(m, n)$ and desired signal $d_2(m, n)$. And the second part consists of the denominator vertical section with input vector $\boldsymbol{\varphi}_2(m, n)$ and desired signal $y_2(m, n)$. Then we show that the effect of the interaction between these two cascaded parts can be replaced by a noise component in the desired signal of each part. On the other-hand, and

in order to reduce the interconnection between the cascaded sections, we here suggest that, at the update iteration k, the signals $y_1(m, n)$, $d_2(m, n)$ and $d_1(m, n)$ should be calculated from the estimated parameters at iteration $k - 1$, while the signal $y_2(m, n)$ and $y(m, n)$ should be calculated from the estimated parameters at iteration $k - 2$. Accordingly, at iteration k, the intermediate signal $y_2(m, n)$ is independent of $\hat{\boldsymbol{\theta}}_1(k - 1)$, that is $e_2(m, n)$ is independent of $\hat{\boldsymbol{\theta}}_1(k - 1)$. In this view, and under the assumption that the adaptation process of the filter coefficients is slow, i.e. the used step size parameters are sufficiently small, stability analysis of the difference equation that describes the parameter-error vector of each part can be carried out as if these two part were independent using the stability robustness theory of perturbed linear system [6] following the same line of arguments as presented in [5].

In the rest of this section, we only discuss the stability of the mean of the parameter-error vector:

$$\mathrm{E}\{\tilde{\boldsymbol{\theta}}_1(k)\} = \mathrm{E}\{\boldsymbol{\theta}_1 - \hat{\boldsymbol{\theta}}_1(k)\} \tag{15}$$

where $\boldsymbol{\theta}_1$ is the ideal parameter vector defined as

$$\boldsymbol{\theta}_1 = [b_1(1), \ldots, b_1(M_1), a(0, 0), \ldots, a(N_1, N_2)]^T \tag{16}$$

Similar mathematical derivation and stability analysis can be applied to the parameter-error vector $\tilde{\boldsymbol{\theta}}_2(k)$ [4].

Stability Analysis of Eq. (15)

Now, before going into the stability analysis of Eq. (15), we will first show that the desired signal $d_2(m, n)$ for the first part of the structure can be decomposed into three components as follows:

$$d_2(m, n) = d_{2_0}(m, n) + \zeta_2(m, n) + v_1(m, n) \tag{17}$$

with $d_{2_0}(m, n)$ represents the noise free stationary desired signal for this part of the structure; $\zeta_2(m, n)$ is a perturbation component related to the fluctuation of the parameter-error vector $\tilde{\boldsymbol{\theta}}_2(k - 1)$.

Using Eq. (2), the regressor vector $\boldsymbol{\varphi}_2(m, n)$ in Eq. (8) can be rewritten as

$$\boldsymbol{\varphi}(m, n) = \boldsymbol{\varphi}_{2_0}(m, n) + \boldsymbol{v}_2(m, n) \tag{18}$$

with

$$\boldsymbol{\varphi}_{2_0}(m, n) = [w(m, n), w(m, n - 1), \ldots, w(m, n - M_2)]^T \tag{19}$$

$$\boldsymbol{v}_2(m, n) = [v(m, n), v(m, n - 1), \ldots, v(m, n - M_2)]^T. \tag{20}$$

From Figure 1, and using Eq. (18), we have

$$
\begin{aligned}
d_2(m, n) &= \hat{\boldsymbol{\theta}}_2(k-1)^T (\boldsymbol{\varphi}_{2_0}(m, n) + \boldsymbol{v}_2(m, n)) \\
&= \boldsymbol{\theta}_2^T \boldsymbol{\varphi}_{2_0}(m, n) - \tilde{\boldsymbol{\theta}}_2^T(k-1) \boldsymbol{\varphi}_{2_0}(m, n) + v_1(m, n) \\
&= \boldsymbol{\theta}_2^T \boldsymbol{\varphi}_{2_0}(m, n) + \zeta_2(m, n) + v_1(m, n)
\end{aligned}
\tag{21}
$$

with $\zeta_2(m, n) = -\tilde{\boldsymbol{\theta}}_2^T(k-1) \boldsymbol{\varphi}_{2_0}(m, n)$.

However, from Eq. (1) we can find that

$$
\boldsymbol{\theta}_1^T \boldsymbol{\varphi}_{1_0}(m, n) = \boldsymbol{\theta}_2^T \boldsymbol{\varphi}_{2_0}(m, n)
\tag{22}
$$

where

$$
\begin{aligned}
\boldsymbol{\varphi}_{1_0}(m, n) &= [d_{2_0}(m-1, n), \ldots, d_{2_0}(m-M_1, n), \\
&\quad\quad u(m, n), \ldots, u(m-N_1, n-N_2)]^T
\end{aligned}
\tag{23}
$$

$$
d_{2_0}(m, n) = \boldsymbol{\theta}_2^T \boldsymbol{\varphi}_{2_0}(m, n).
\tag{24}
$$

Substituting Eq. (22) in Eq. (21) we arrive at Eq. (17) with $d_{2_0}(m, n) = \boldsymbol{\theta}_1^T \boldsymbol{\varphi}_{1_0}(m, n)$.

Now, substituting Eq. (12) in Eq. (15) and using Eqs. (3) and (17) we get

$$
\mathrm{E}\{\tilde{\boldsymbol{\theta}}_1(k)\} = [A_1 + B_1(m, n)]\mathrm{E}\{\tilde{\boldsymbol{\theta}}_1(k-1)\} + A_{v_1} + A_{\zeta_2}
\tag{25}
$$

where

$$
A_1 = I - \mu_1 \boldsymbol{R}_{\varphi_1 \varphi_1}
\tag{26}
$$

$$
B_1(m, n) = \mu_1 \mathrm{E}\{\tau_1(m, n)\boldsymbol{\varphi}_1(m, n)\varepsilon_2^T(m, n)\}
\tag{27}
$$

$$
A_{v_1} = \mu_1 \mathrm{E}\{[\boldsymbol{b}_1^T \boldsymbol{v}_1(m, n) - v_1(m, n)][\boldsymbol{\varphi}_1(m, n) - \tau_1(m, n)\varepsilon_2(m, n)]\}
\tag{28}
$$

$$
A_{\zeta_2} = -\mu_1 \mathrm{E}\{[1, -\boldsymbol{b}_1^T]\boldsymbol{\zeta}_2(m, n)[\boldsymbol{\varphi}_1(m, n) - \tau_1(m, n)\varepsilon_2(m, n)]\}
\tag{29}
$$

with

$$
\boldsymbol{R}_{\varphi_1 \varphi_1} = \mathrm{E}\{\boldsymbol{\varphi}_1(m, n)\boldsymbol{\varphi}_1^T(m, n)\}
\tag{30}
$$

$$
\boldsymbol{b}_1 = [b_1(1), \ldots, b_1(M_1)]^T
\tag{31}
$$

$$
\boldsymbol{v}_1(m, n) = [v_1(m-1, n), \ldots, v_1(m-M_1, n)]^T
\tag{32}
$$

$$
\boldsymbol{\zeta}_2(m, n) = [\zeta_2(m, n), \zeta_2(m-1, n), \ldots, \zeta_2(m-M_1, n)]^T.
\tag{33}
$$

Note that we have assumed that the parameter vector $\hat{\boldsymbol{\theta}}_1(k-1)$ is independent of the regressor vector $\boldsymbol{\varphi}_1(m, n)$. This assumption is referred to as the independence assumption in adaptive filtering literature [7] and is well justified for sufficiently small step size parameters.

The difference equation (25) has two forcing terms. The first one A_{ζ_2} is related to the error in estimating the parameters of the second part of the cascade structure $\tilde{\boldsymbol{\theta}}_2(k-1)$. And the second A_{v_1} is due to the colored measurement noise $v_1(m, n)$. For $\tau_1(m, n) = 0$, these two forcing terms cause the adaptive algorithm [2] to converge to biased solution.

Table 1. Parameter estimates for *Example 1* (30 runs).

	$a(1, 0)$	$a(0, 1)$	$a(1, 1)$	$b_1(1)$	$b_1(2)$	$b_2(1)$	$b_2(2)$
True values	0.8	−0.5	−0.4	1.2	−0.36	−0.9	−0.2
Ref. [2]	1.3208	−0.4717	−0.6225	0.6844	0.0980	−0.9174	−0.2346
2DBRA	0.7967	−0.5028	−0.4063	1.2050	−0.3591	−0.8981	−0.1994

In the Appendix we discuss the conditions that should be imposed on the step size parameter μ_1 and the constant α_1 to ensure the stability of the difference equation (25). We show that the forcing term A_{v_1} will approach zero as the scaling factor $\tau_1(m, n) \to 1$. On the other-hand, the output error feedback in the second part of the structure will be useful to reduce the amount of bias caused by the forcing term A_{ζ_2}.

4. Experimental Results

Example 1: Noisy desired signal (white measurement noise). In this example, the 2-D adaptive algorithm [2] and the 2DBRA are applied to the system identification experiment. A 2-D zero mean white Gaussian signal of unit variance and size 256 by 256 is used for the input signal $u(m, n)$. And a zero mean, unit variance Gaussian noise which is independent of the input signal is used for the additive noise $v(m, n)$. The process $w(m, n)$ is generated by filtering the input signal $u(m, n)$ with the separable denominator 2-D IIR filter:

$$H\left(q_1^{-1}, q_2^{-1}\right) = \frac{1 + 0.8q_1^{-1} - 0.5q_2^{-1} - 0.4q_1^{-1}q_2^{-1}}{(1 - 1.2q_1^{-1} + 0.36q_1^{-2})(1 + 0.9q_2^{-1} + 0.2q_2^{-2})}. \tag{34}$$

Table 1 shows the obtained parameter estimates for $\mu_1 = 0.0012$, $\mu_2 = 0.0008$, and $\alpha_1 = \alpha_2 = 0.5$.

Experimental results have shown that the amount of the reduction in the bias caused by the interaction between the cascaded sections depends on the values of the used step size parameters μ_1 and μ_2. For sufficiently small step size values, significant reduction in the bias caused by both the measurement noise and the cascade interaction can be obtained. This reduction in the bias is achieved, however, at the expense of moderate increase in the computational load of the 2DBRA over the algorithm [2]. This increase is due to the requirement of adjusting the time varying scaling factors $\tau_1(m, n)$ and $\tau_2(m, n)$ at each iteration.

Example 2: Noisy desired signal (colored measurement noise). In this example we aim to compare the performances of the proposed 2DBRA with that of the 2-D LMS algorithm [2] in the colored measurement noise scenario. We apply both algorithms to the system identification experiment. The following transfer function is used for the unknown system [8]:

$$H\left(q_1^{-1}, q_2^{-1}\right) = \frac{1 + q_1^{-1} + 2q_2^{-1} + 3q_1^{-1}q_2^{-1}}{1 - 0.25q_1^{-1} - 0.5q_2^{-1} + 0.125q_1^{-1}q_2^{-1}}. \tag{35}$$
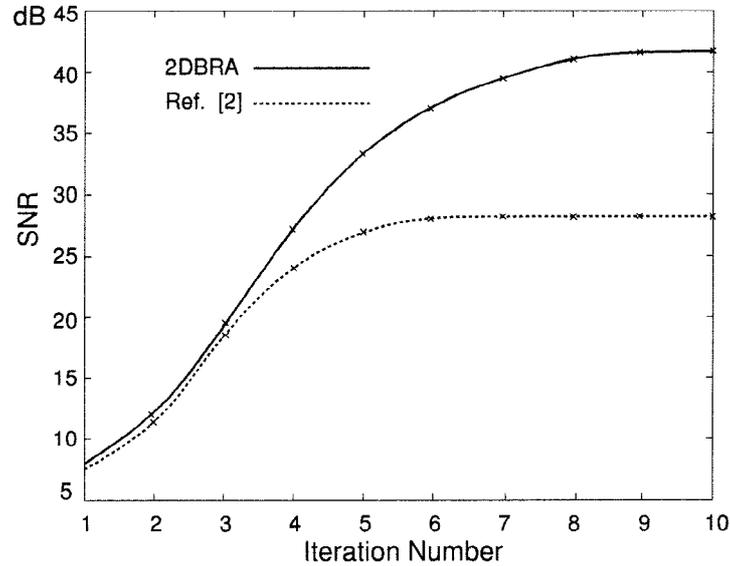
Figure 2. The improvement of the SNR of the enhanced image through consecutive passes in *Example* 2.

A 2-D white Gaussian noise of zero mean and 104 variance is used for the input signal $u(m, n)$. And the "Mandrill" image of mean 129.1378, variance $\sigma_v^2 = 1749.8$, and size 256 columns by 256 rows, is used for the measurement noise $v(m, n)$. The image is processed row by row repeatedly such that the values of the estimated parameters at the end of one pass are used as the parameter initial values at the beginning of the next pass. The error signal $e_0(m, n) = d(m, n) - y(m, n)$ gives the enhanced image. The variance of the noise left in the enhanced image is calculated by subtracting the noiseless image from it and then computing the variance.

Figure 2 shows the improvement in the Signal to Noise Ratio (SNR) of the enhanced image through successive passes using the algorithm [2] and the 2DBRA where SNR value is calculated by

$$SNR = 10 \ \log \left(\frac{\sigma_v^2}{\sigma_n^2} \right) \tag{36}$$

with σ_n^2 denotes the variance of the noise left in the enhanced image. The SNR value of the initial noisy image is -7.711. As this figure indicates, for low SNR, the 2DBRA works almost without the bias removal mechanism as the algorithm [2]. As the SNR increases gradually, the effectiveness of the output-error-feedback becomes very clear. Figures 3 and 4 show the original and the noisy image "Mandrill" respectively. Figures 5 and 6 show the enhanced images at the 10th pass using algorithm [2] and the 2DBRA respectively.

In both of the experiments presented here, stability monitoring was not required. It has been observed that, for sufficiently small step size parameters, whenever the poles of the
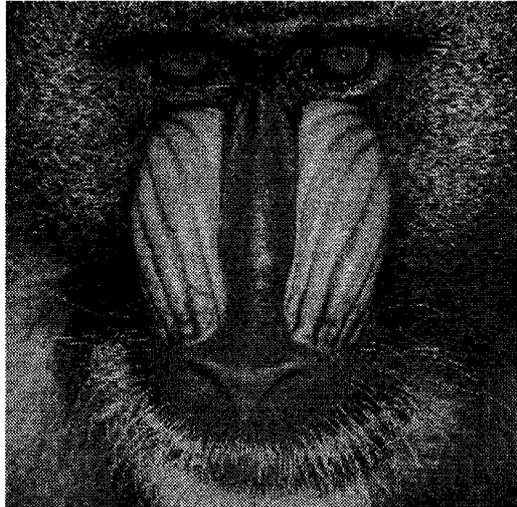
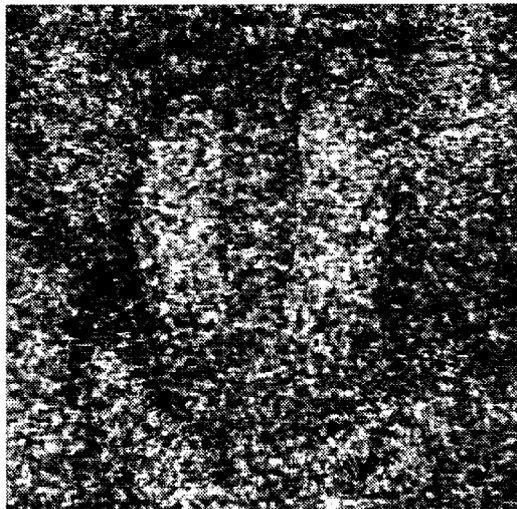*Figure 3.* The original image "Mandrill" used in *Example* 2.

Figure 4. The noisy image "Mandrill" used in *Example* 2.

adaptive filters start to immigrate outside the unit circle, the output error of each section increases suddenly and consequently the scaling factors $\tau_1(m, n)$ and $\tau_2(m, n)$ decreases; the adaptive algorithm works without output-error feedback and is able to draw the poles back to the unit circle.

Figure 5. The enhanced image at the 10th pass using algorithm [2], $\mu_1 = \mu_2 = 7.6894 \times 10^{-8}$.

Figure 6. The enhanced image at the 10th pass using the 2DBRA, $\mu_1 = \mu_2 = 7.6894 \times 10^{-8}, \alpha_1 = \alpha_2 = 0.7$.

5. Conclusion

The bias removal algorithm for 2-D equation error adaptive IIR filters has been presented. The filter structure in the proposed algorithm is based on backpropagating the desired signal

through a cascade of the denominator vertical and horizontal sections. The key idea in the proposed algorithm is to use a scaled value of the output error of each of the cascade sections to counteract the effect of the measurement noise embedded in the regressor of the update procedure of that section. I/O stability analysis has been carried out. It has been shown that the proposed algorithm remains stable and the effect of the measurement noise can be significantly suppressed under general conditions imposed on the values of the used step sizes and scaling factors. Image enhancement and 2-D system identification experimental results have been presented to support the effectiveness of the proposed 2DBRA algorithm.

Appendix: Stability Analysis of the Difference Equation (25)

i) *Stability Analysis of the Autonomous Part of Eq. (25)*

The autonomous perturbed system

$$\text{E}\{\tilde{\boldsymbol{\theta}}_1(k)\} = [A_1 + B_1(m, n)]\text{E}\{\tilde{\boldsymbol{\theta}}_1(k - 1)\}, \tag{37}$$

is exponentially and asymptotically stable if [6]

1. *All the eigenvalues of the stability matrix A_1 are within the unit circle.* Such condition is satisfied if

$$\mu_1 \leq \frac{2\sigma_1}{\text{tr}(\boldsymbol{R}_{\varphi_1\varphi_1})}, \quad 0 < \sigma_1 \leq 1. \tag{38}$$

Then we can define the state transition matrix

$$\Phi_1(k) = A_1^k, \quad k > 0 \tag{39}$$

that satisfies

$$\|\Phi_1(k)\| \leq c_1\beta_1^k, \quad c_1 > 0, \quad \beta_1 \in [0, 1]. \tag{40}$$

2. *The perturbation term $B_1(m, n)$ satisfies.*

$$0 \leq \beta_1 + c_1\|B_1(m, n)\| \leq 1. \tag{41}$$

From Eq. (27) we have

$$
\begin{aligned}
\|B_1(m, n)\| &= \|\text{E}\left\{\tau_1(m, n)\mu_1\varphi_1(m, n)\varepsilon_2^T(m, n)\right\}\| \\
&\leq \alpha_1\text{E}\left\{\frac{\|\varphi_1(m, n)\|}{\|\varepsilon_2(m, n)\|}\frac{2\sigma_1\|\varphi_1(m, n)\|}{\text{tr}(\boldsymbol{R}_{\varphi_1\varphi_1})}\|\varepsilon_2(m, n)\|\right\} \\
&\leq 2\sigma_1\alpha_1. \tag{42}
\end{aligned}
$$

Thus, the value of α_1 should satisfy

$$0 \leq \alpha_1 \leq \frac{1 - \beta_1}{2\sigma_1 c_1}. \tag{43}$$

ii) *The Boundedness of the Forcing Term A_{v_1}*

Making use of Eq. (17) in Eq. (28), invoking the assumption that the measurement noise $v(m, n)$ is independent of $u(m, n)$ and $w(m, n)$, and providing that the poles of the adaptive filter's denominator horizontal section are inside the unit circle, we can find that

$$
\begin{aligned}
\|A_{v_1}\| &= \mu_1 \|\mathrm{E}\{[\boldsymbol{b}_1^T \boldsymbol{v}_1(m, n) - v_1(m, n)][v_1(m, n) - \tau_1(m, n)v_1(m, n)]\}\| \\
&= \mu_1 \|[1 - \tau_1(m, n)][\boldsymbol{R}_{v_1 v_1} \boldsymbol{b}_1 - \mathrm{E}\{v_1(m, n)\boldsymbol{v}_1(m, n)\}]\| \\
&\leq \mu_1 \|\boldsymbol{R}_{v_1 v_1} \boldsymbol{b}_1 - \mathrm{E}\{v_1(m, n)\boldsymbol{v}_1(m, n)\}\|
\end{aligned} \tag{44}
$$

with $\boldsymbol{R}_{v_1 v_1} = \mathrm{E}\{\boldsymbol{v}_1(m, n)\boldsymbol{v}_1^T(m, n)\}$. Hence, the forcing term A_{v_1} is bounded, and $\|A_{v_1}\|$ will approach zero as $\tau_1(m, n) \to 1$.

iii) *The Boundedness of the Forcing Term A_{ζ_2}*

From Eq. (29) we have

$$
\begin{aligned}
\|A_{\zeta_2}\| &= \mu_1 \|\mathrm{E}\{[1, \ -\boldsymbol{b}_1]\zeta_2(m, n)[\boldsymbol{\varphi}_1(m, n) - \tau_1(m, n)\varepsilon_2(m, n)]\}\| \\
&\leq \mu_1 \left(\sum_{i=0}^{M_1} b_1^2(i) \right)^{1/2} \mathrm{E}\{\|\boldsymbol{\zeta}_2(m, n)\| \|[\boldsymbol{\varphi}_1(m, n) - \tau_1(m, n)\varepsilon_2(m, n)]\|\}.
\end{aligned} \tag{45}
$$

Equation (45) states that the norm of the forcing term A_{ζ_2} is bounded providing that the error signal $\zeta_2(m, n)$ has finite variance. The boundedness of the error $\zeta_2(m, n)$ is guaranteed if the step size μ_2, used in the update procedure of the denominator vertical section, satisfies the necessary condition for the convergence of the LMS FIR filter $\hat{B}_2(q_2^{-1})$ in the variance which is given by [7]

$$\mu_2 < \frac{2}{3\mathrm{tr}(\boldsymbol{R}_{\varphi_2 \varphi_2})} \tag{46}$$

with $\boldsymbol{R}_{\varphi_2 \varphi_2} = \mathrm{E}\{\boldsymbol{\varphi}_2(m, n)\boldsymbol{\varphi}_2^T(m, n)\}$. In order to reduce the influence of the forcing term A_{ζ_2}, a sufficiently small step size μ_2 that guarantees small variance for $\tilde{\boldsymbol{\theta}}_2(k)$, and hence for $\zeta_2(m, n)$, should be used.

References

1. F. X. Y. Gao and W. M. Snelgrove, "An Adaptive Backpropagation Cascade IIR Filter," *IEEE Trans. Circuits Syst. II*, vol. CAS-39, no. 9, September 1992, pp. 606–610.

2. K. Toshima, M. Ohki, X. Zhou, and H. Hashiguchi, "2-D Backpropagation Cascade Adaptive Recursive Filter with the Separable Denominator Function," *Third International Symposium on Consumer Electronics*, vol. 2, November 1994, pp. 441–446.

3. D. E. Dudgeon and R. Mersereau, *Multidimensional Digital Signal Processing*, Prentice Hall, 1994.

4. M. Shadaydeh and M. Kawamata, "Bias Removal Algorithm for 2-D Equation Error Adaptive IIR Filters," *Proc. 5th IEEE International Workshop on Intelligent Signal Processing and Communication Systems*, November 1997, Malaysia.

5. J. N. Lin and R. Unbehauen, "Bias-Remedy Least Mean Square Equation Error Algorithm for IIR Parameter Recursive Estimation," *IEEE Trans. Signal Processing*, vol. 40, no. 1, January 1992, pp. 62–69.

6. Z. Gajiç and M. T. J. Qureshi, *Lyapunov Matrix Equation in System Stability and Control*, Academic Press, 1995.

7. V. Solo and X. Kong, *Adaptive Signal Processing Algorithms*, NJ: Prentice-Hall, 1995.

8. A. C. Tan, S. T. Chen, and S. Basu, "Image Filtering Using Hyperstable Adaptive Algorithms," *IEEE Trans. Circuits Syst. II*, vol. CAS-44, no. 5, May 1997, pp. 358–370.