PERFORMANCE ANALYSIS OF ADAPTIVE IIR NOTCH FILTERS BASED ON LEAST MEAN P-POWER ERROR CRITERION

Maha Shadaydeh *Masayuki Kawamata*

Electronic Engineering Department, Graduate School of Engineering
Tohoku University, Sendai 980-8579, Japan.
maha@mk.ecei.tohoku.ac.jp, kawamata@mk.ecei.tohoku.ac.jp

ABSTRACT

In this paper, we present the steady state analysis of adaptive IIR notch filters based on the least mean p-power error criterion. We consider the cases when the sinusoidal signal is contaminated with white Gaussian noise and $p = 3, 4$. We first derive two difference equations for the convergence of the mean and the Mean Square Error (MSE) of the adaptive filter's notch coefficient, and then give the steady state estimation bias and MSE. Stability conditions on the step size value are also derived. Simulation experiments are presented to confirm the validity of the obtained analytical results. It is shown that the notch coefficient steady state bias of the $p-$power algorithm for small step size values is independent of the step size value and is equal for $p = 1, 2, 3$ and 4. However, for larger step size values, the p-power algorithm with $p = 3$ provides the best performance in term of the MSE.

1. INTRODUCTION

Adaptive IIR notch filters have been successfully used for detecting sinusoidal signals in wide-band noise. So far, several adaptive IIR notch filtering algorithms based on least Mean Square Error (MSE) criteria have been proposed [1]- [3]. However, MSE criteria do not always provide the best performance, and accordingly there has been increased interest in developing adaptive algorithms based on L_p normed minimization. So far several L_p norm based adaptive IIR notch filtering algorithms have been proposed, such as the Sign Algorithm (SA) [4], [5] and the p-power algorithm [6]. However, the question to be asked here is: for which value of p does the $p-$power algorithm provide the best performance? The answer to this question, as simulation experiments show, depends on the nature of additive noise. The SA algorithm seems to provide the best performance when the sinusoidal signal is contaminated with impulsive noise [5]. However, for the case when the additive noise is Gaussian, Pei et al. [6] have shown by intensive simulations that the performance of the $p-$power algorithm for $p = 3$ is better than that of the LMS algorithm (i.e. $p = 2$) and the SA (i.e. $p = 1$). The performance analysis for SA and LMS algorithm have been intensively studied in the literature [1]- [3], [7]. However for $p > 2$, no performance analysis has been presented so far.

In this paper, we present the steady state analysis of the p-power algorithm [6] for the cases when $p = 3$ and $p = 4$, and the additive noise is white Gaussian. After a short review of the p-power algorithm, two difference equations for the convergence of the mean and MSE are derived and stability bounds on the step size value are discussed. Closed form expressions for the steady state bias and MSE are then concluded. Simulation experiments that confirm the obtained analytical results are presented with some comparison remarks on the performance of the p-power algorithm for different values of p.

2. THE P-POWER ALGORITHM

In this paper the second-order IIR notch filter with constrained poles and zeros [8] is considered. Its transfer function is given by

$$H(z) = \frac{1 + az^{-1} + z^{-2}}{1 + \rho a z^{-1} + \rho^2 z^{-2}} \tag{1}$$

where ρ is the pole radius of the adaptive filter which is restricted to the range $[0, 1)$ to insure stability of the IIR filter. The parameter a in (1) is the filter notch coefficient; its true value is calculated by $a_0 = -2\cos\omega_0$, where ω_0 is the frequency of the input sinusoidal signal

$$x(n) = A\cos(\omega_0 n + \theta) + v(n). \tag{2}$$

The additive noise $v(n)$ in (2) is assumed to be zero mean white Gaussian noise with variance σ_v^2; A and θ are the unknown signal amplitude and phase.

The $p-$power algorithm [6] updates the notch coefficient a such that the mean $p-$power of the notch filter's output signal $e(n)$, that is $\mathrm{E}(|e(n)|^p)$, is minimized. Accordingly, using the steepest descent algorithm, the update

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equation of the filter's notch coefficient estimation error $\delta_a(n) = \hat{a}(n) - a_0$ is given by

$$\delta_a(n+1) = \delta_a(n) - \mu_p e^{p-1}(n)\ \mathrm{sign}(e(n))\ s(n) \quad (3)$$

for p odd, and by

$$\delta_a(n+1) = \delta_a(n) - \mu_p\ e^{p-1}(n) s(n) \quad (4)$$

for p even. μ_p is the step size value, and $s(n)$ is the gradient signal calculated by

$$s(n) = \frac{\partial e(n)}{\partial \hat{a}} \approx x(n-1) - \rho e(n-1). \quad (5)$$

3. PERFORMANCE ANALYSIS

At the steady state, the filter's notch coefficient $\hat{a}(n)$ becomes close enough to its true value a_0. Thus, using Taylor series expansion of the notch filter transfer function (1) in the vicinity of a_0, the output and gradient signals can be calculated by [3]

$$\begin{aligned} e(n) &= AB\cos(\omega_0 n + \theta - \phi)\delta_a(n) & (6)\\ &\quad -\rho AB^2 \cos(\omega_0 n + \theta - 2\phi)\delta_a^2(n) + v_1(n) \\ s(n) &= A\cos(\omega_0 n + \theta - \omega_0) & (7)\\ &\quad -\rho AB\cos(\omega_0 n + \theta - \omega_0 - \phi)\delta_a(n) \\ &\quad +\rho AB^2\cos(\omega_0 n + \theta - \omega_0 - 2\phi)\delta_a^2(n) + v_2(n) \end{aligned}$$

where $v_1(n)$ and $v_2(n)$ are the additive noise in the filter's output and gradient signal respectively. ϕ and B are defined as

$$B = \frac{1}{(1-\rho)\sqrt{(1+\rho)^2 - 4\rho\cos^2\omega_0}} \quad (8)$$

$$\phi = \begin{cases} \phi_0, & \omega_0 \le \frac{\pi}{2} \\ \phi_0 + \pi, & \omega_0 > \frac{\pi}{2} \end{cases} ;\phi_0 = \tan^{-1}\frac{(1+\rho)\sin(\omega_0)}{(1-\rho)\cos(\omega_0)}.$$

In our analysis for $p = 3$, to handle the sign function in the update equation (3), we use similar approach to that used in the analysis of the SA [7] which is based on the assumption that the output signal $e(n)$ is Gaussian distributed with mean value μ_e and variance σ_e^2 that can be calculated directly from (6), and that $e(n)$ and $\delta_a(n)$ are jointly Gaussian distributed. This assumption has been tested in [7] and proved to hold as long as the noise $v(n)$ is white and not necessarily Gaussian.

3.1. Convergence of the Mean

Substituting (6) and (7) in (3)((4)), applying the expectation operator E, and after long mathematical work, we can get the following difference equation for the convergence of the mean:

$$\begin{aligned} \mathrm{E}[\delta_a(n+1)] &= (1 - \mu_p A_{p,1})\,\mathrm{E}[\delta_a(n)] & (9)\\ &\quad -\mu_p B_{p,1}\,\mathrm{E}[\delta_a^2(n)] - \mu_p C_{p,1}, \quad p = 3, 4, \end{aligned}$$

where for $p = 3$,

$$\begin{aligned} A_{3,1} &= \frac{3}{\sqrt{2\pi}}\sigma_{v_1} A^2 B\cos(\omega_0 - \phi) \\ B_{3,1} &= -\frac{3}{\sqrt{2\pi}}\sigma_{v_1}\rho A^2 B^2(\cos(\omega_0 - 2\phi) + \cos(\omega_0)) \\ &\quad + \frac{3}{\sqrt{2\pi}} A^2 B^2 \frac{R_{1,2}}{\sigma_{v_1}} \\ C_{3,1} &= \frac{6}{\sqrt{2\pi}}\sigma_{v_1} R_{1,2}, & (10) \end{aligned}$$

and for $p = 4$,

$$\begin{aligned} A_{4,1} &= 1.5\sigma_{v_1}^2 A^2 B\cos(\omega_0 - \phi) \\ B_{4,1} &= -1.5\sigma_{v_1}^2 \rho A^2 B^2 (\cos(\omega_0 - 2\phi) + \cos(\omega_0)) \\ &\quad +1.5A^2B^2R_{1,2} \\ C_{4,1} &= 3\sigma_{v_1}^2 R_{1,2}. & (11) \end{aligned}$$

$\sigma_{v_1}^2$, $\sigma_{v_2}^2$, and $R_{1,2}$ are respectively the variance of $v_1(n)$, the variance of $v_2(n)$, and the correlation between $v_1(n)$ and $v_2(n)$, and can be calculated using the theory of residues [3].

In the calculation of (9) and (12) (presented in the following subsection), we need to go through the calculations of many terms of the general form $\mathrm{E}[\delta_a^m(n)\mathrm{sign}(e(n))v_1^l(n)v_2^k(n)]$ for $p = 3$ or of the form $\mathrm{E}[\delta_a^m(n)\ v_1^l(n)\ v_2^k(n)]$ for $p = 3, 4$, where $m = 0, \cdots, 16(p-1)$, $l = 0, \cdots, 2(p-1)$ and $k = 0, 1, 2$. These terms are calculated using the Gaussian factoring theorem, the relations between higher order cumulants of random signals and their moments, and the property that higher order cumulants of Gaussian signals equal zero. The terms of $\delta_a^m(n)$, with $m \ge 3$ are ignored. It is also assumed that the estimation error $\delta_a(n)$ is uncorrelated with the noise signals $v_1(n)$ and $v_2(n)$. For $p = 3$, the joint moments of $\mathrm{sign}(e(n))$ and each of $\delta_a(n)$, $v_1(n)$ and $v_2(n)$ are calculated using the Gaussian probability distribution function of the output signal $e(n)$ [3]. Calculation details are omitted here due to space limitation.

3.2. Convergence of the MSE

Squaring both sides of (3)((4)), using (6) and(7), and then averaging, we can after long calculations, derive the following difference equation for the convergence of the MSE:

$$\begin{aligned} \mathrm{E}[\delta_a^2(n+1)] &= (1 - \mu_p B_{p,2})\,\mathrm{E}[\delta_a^2(n)] & (12)\\ &\quad -\mu_p A_{p,2}\,\mathrm{E}[\delta_a(n)] + \mu_p^2 C_{p,2}, \quad p = 3, 4 \end{aligned}$$

where for $p = 3$,

$$\begin{aligned} A_{3,2} = \frac{12}{\sqrt{2\pi}}\sigma_{v_1}R_{1,2} - \mu_p\,\big(&12A^2BR_{1,2}\sigma_{v_1}^2\cos(\omega_0 - \phi) \\ &-3\rho A^2 B\sigma_{v_1}^4\cos(\phi)\big) \end{aligned}$$

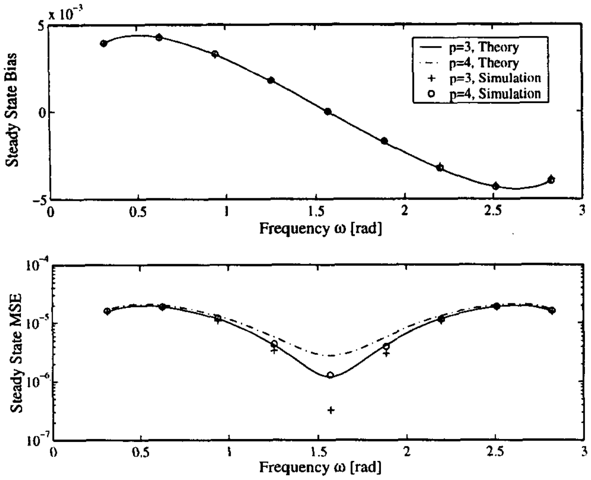

Figure 1: Comparison between theoretical and simulated steady state estimation bias and MSE versus the sinusoidal frequency of the input signal ω_0 ($\mu_p = 0.00005, \rho = 0.9, A = \sqrt{2}$, SNR=5).

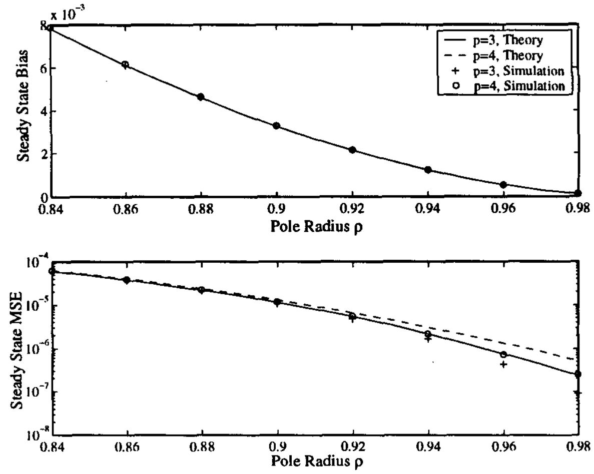

Figure 2: Comparison between theoretical and simulated steady state bias and MSE versus the pole radius ρ ($\mu_p = 0.00005, \omega_0 = 0.3\pi, A = \sqrt{2}$, SNR=5).

$$
\begin{aligned}
B_{3,2} &= \frac{6}{\sqrt{2\pi}} A^2 B \sigma_{v_1} \cos(\omega_0 - \phi) \\
&\quad -\mu_p \left(1.5 A^4 B^2 \sigma_{v_1}^2 (0.5 + \cos^2(\omega_0 - \phi))\right. \\
&\quad +3A^2B^2\sigma_{v_1}^2\sigma_{v_2}^2 + 6A^2B^2R_{1,2}^2 \\
&\quad +3\rho A^2 B^2 \sigma_{v_1}^4 (\cos(2\phi) + 0.5\rho) \\
&\quad \left. -12R_{1,2}\rho A^2 B^2 \sigma_{v_1}^2 (\cos(\omega_0 - 2\phi) + \cos(\omega_0))\right) \\
C_{3,2} &= \frac{3}{2} A^2 \sigma_{v_1}^4 + 3\sigma_{v_2}^2 \sigma_{v_1}^4 + 12 R_{1,2}^2 \sigma_{v_1}^2, \qquad (13)
\end{aligned}
$$

and for $p = 4$,

$$
\begin{aligned}
A_{4,2} &= 6\sigma_{v_1}^2 R_{1,2}^2 - \mu_p \left(90 A^2 B \sigma_{v_1}^4 R_{1,2} \cos(\omega_0 - \phi)\right. \\
&\quad \left. - 15\rho\sigma_{v_1}^6 A^2 B \cos(\phi)\right) \\
B_{4,2} &= 3A^2\sigma_{v_1}^2 B \cos(\omega_0 - \phi) \\
&\quad -\mu_p \left((45/4) A^4 B^2 \sigma_{v_1}^4 (0.5 + \cos^2(\omega_0 - \phi))\right. \\
&\quad +(15/2) A^2 B^2 \sigma_{v_1}^6 \rho^2 - 15\rho A^2 B \sigma_{v_1}^6 \cos(\phi) \\
&\quad -90\rho A^2 B^2 \sigma_{v_1}^4 R_{1,2} (\cos(\omega_0) + \cos(\omega_0 - 2\phi)) \\
&\quad \left. +(45/2) A^2 B^2 \sigma_{v_2}^2 \sigma_{v_1}^4\right) \\
C_{4,2} &= \frac{15}{2} A^2 \sigma_{v_1}^6 + 15\sigma_{v_2}^2 \sigma_{v_1}^6. \qquad (14)
\end{aligned}
$$

3.3. Stability Bounds

Stability bounds on the step size value can now be easily obtained. In fact if the influence of the the second term in (9) is ignored, the sufficient condition for the convergence of the mean is then given by

$$|1 - \mu_p A_{p,1}| < 1, \qquad p = 3, 4. \qquad (15)$$

Providing that (15) holds, the sufficient condition for the convergence of the MSE can then be deduced from (12) as

$$|1 - \mu_p B_{p,2}| < 1, \qquad p = 3, 4. \qquad (16)$$

3.4. Steady State Estimation Bias and MSE

At the steady state, we have

$$
\begin{aligned}
\mathrm{E}[\delta_a^2(n+1)]_{n\to\infty} &= \mathrm{E}[\delta_a^2(n)]_{n\to\infty} = \mathrm{E}[\delta_a^2(\infty)] \\
\mathrm{E}[\delta_a(n+1)]_{n\to\infty} &= \mathrm{E}[\delta_a(n)]_{n\to\infty} = \mathrm{E}[\delta_a(\infty)]. \qquad (17)
\end{aligned}
$$

Using (17) in (9) and (12), solving the resulting two equations simultaneously, the following closed form expressions for the steady state bias and MSE can be obtained

$$\mathrm{E}[\delta_a(\infty)] = \frac{B_{p,2}C_{p,1} + \mu_p B_{p,1} C_{p,2}}{A_{p,2}B_{p,1} - A_{p,1}B_{p,2}}, p = 3, 4 \qquad (18)$$

$$\mathrm{E}[\delta_a^2(\infty)] = \frac{A_{p,2}C_{p,1} + \mu_p A_{p,1} C_{p,2}}{A_{p,1}B_{p,2} - A_{p,2}B_{p,1}}, p = 3, 4. \qquad (19)$$

3.5. Simulation Results

To confirm the obtained analytical results, we have conducted several experiments. Figure 1 shows comparison between simulated and theoretical steady state bias and MSE versus the sinusoidal signal frequency ω_0 for $p = 3, 4$.

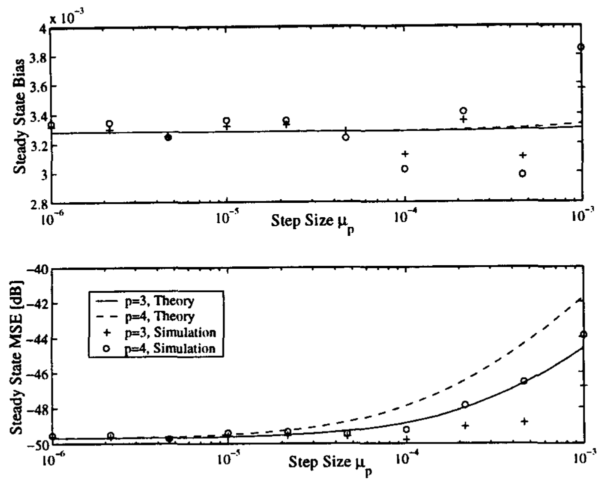

Figure 3: Comparison between theoretical and simulated steady state bias and MSE versus the step size value μ_p ($\rho = 0.9, \omega_0 = 0.3\pi, A = \sqrt{2}$, SNR=5).

It can be observed that the theoretical results match the simulations very well except in the the neighborhood of $\omega_0 = 0.5\pi$. Figure 2 shows comparison between simulated and theoretical steady state bias and MSE versus the pole radius ρ for $p = 3, 4$. As expected, the bias and MSE decrease as the pole radius ρ increases. Figure 3 shows comparison between simulated and theoretical steady state bias and MSE versus the step size value μ_p. This figure indicates that the $p - power$ algorithm results in similar steady state bias for both $p = 3$ and $p = 4$ for sufficiently small step size values. In fact, for small step size values the second term in the numerator of (18) can be neglected and it can be verified that the p-power algorithm has similar expression for the steady state bias for both $p = 3$ and $p = 4$ which is independent of the step size value and is mainly due to the correlation $R_{1,2} = \mathrm{E}[v_1(n)v_2(n)]$. Interestingly, it is similar to the bias expression for the SA (i.e. $p = 1$) [7] and the LMS (i.e. $p = 2$) [3]. However, as it can be observed from Figure 3, the $p-$power algorithm with $p = 3$ performs better than that with $p = 4$ for larger step size values. It has been observed through many experimental results that the convergence speed of the p-power algorithm with $p = 3$ is better than that with $p = 1, 2, 4$. Detailed comparison of the performance of this algorithm for different values of p is out of scope of this paper and will be presented later on.

4. CONCLUSION

This paper presented steady state analysis for constrained adaptive IIR notch filters based on least mean p-power error criterion for the cases when $p = 3, 4$ and the sinusoidal signal is contaminated with white Gaussian noise. Closed form expressions for the steady state estimation bias and MSE have been derived and step size stability bounds have been presented. Simulation results confirm the analytical results. It has been shown that for small step size values, the p-power algorithm produces similar estimation bias for all values of p. However, for larger step size values, the $p-$power algorithm with $p = 3$ performs better than that with $p = 4$.

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